

Homoclinic orbits for a class of nonperiodic Hamiltonian systems with some twisted conditions

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Abstract: In this paper, by the Maslov index theory, we will study the existence and multiplicity of homoclinic orbits for a class of asymptotically linear nonperiodic Hamiltonian systems with some twisted conditions on the Hamiltonian functions.

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1 Introduction and main results

Consider the following first order non-autonomous Hamiltonian systems

$$\dot{z} = JH_z(t, z), \quad (\text{HS})$$

where $z : \mathbb{R} \rightarrow \mathbb{R}^{2N}$, $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$, $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ and $\nabla_z H(t, z)$ denotes the gradient of $H(t, z)$ with respect to z . As usual we say that a nonzero solution $z(t)$ of (HS) is homoclinic (to 0) if $z(t) \rightarrow 0$ as $|t| \rightarrow \infty$.

As a special case of dynamical systems, Hamiltonian systems are very important in the study of gas dynamics, fluid mechanics, relativistic mechanics and nuclear physics. While it is well known that homoclinic solutions play an important role in analyzing the chaos of Hamiltonian systems. If a system has the transversely intersected homoclinic solutions, then it must be chaotic. If it has the smoothly connected homoclinic solutions,

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then it cannot stand the perturbation, its perturbed system probably produces chaotic phenomena. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic solutions of Hamiltonian systems emanating from 0.

In the last years, the existence and multiplicity of homoclinic orbits for the first order system (HS) were studied extensively by means of critical point theory, and many results were obtained under the assumption that $H(t, z)$ depends periodically on t (see, e.g., [5, 11–13, 17, 20, 27–30]). Without assumptions of periodicity the problem is quite different in nature and there is not much work done so far. To the best of our knowledge, the authors in [14] firstly obtained the existence of homoclinic orbits for a class of first order systems without any periodicity on the Hamiltonian function. After this, there were a few papers dealing with the the existence and multiplicity of homoclinic orbits for the first order system (HS) in this situation (see, e.g., [15, 16, 21]).

In the present paper, with the Maslov index theory of homoclinic orbits introduced by Chen and Hu in [10], we will study the existence and multiplicity of homoclinic orbits for (HS) without any periodicity on the Hamiltonian function. To the best of the author's knowledge, the Maslov index theory of homoclinic orbits is the first time to be used to study the existence of homoclinic solutions. We are mainly interested in the Hamiltonian functions of the form

$$H(t, z) = -L(t)z \cdot z + R(t, z), \quad (1.1)$$

where L is an $2N \times 2N$ symmetric matrix valued function. We assume that $(L_1)L \in C(\mathbb{R}, \mathbb{R}^{N^2})$, and there are $\alpha, c > 0, t_0 \geq 0$ and a constant matrix P , satisfying

$$PL(t) - c|t|^\alpha I_{2N} \geq 0, \quad \forall |t| \geq t_0,$$

where I_{2N} is the identity map on \mathbb{R}^{2N} and for a $2N \times 2N$ matrix M , we say $M \geq 0$ if and only if

$$\inf_{\xi \in \mathbb{R}^{2N}, |\xi|=1} M\xi \cdot \xi \geq 0.$$

In (L_1) , if $P = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}$, then (L_1) is similar to the condition (R_0) in [15]. But the restrictions on $R(t, z)$ will be different from [15], and we will give some examples in Remark 1.5. If $P = \pm I_{2N}$ or $P = \begin{pmatrix} I_{N+m} & 0 \\ 0 & -I_{N-m} \end{pmatrix}$ in condition (L_1) , for examples, it's quite different from the existing results as authors known. In short, condition (L_1) means that the eigenvalues of $L(t)$ will tend to $\pm\infty$ with the speed no less than $|t|^\alpha$. But (L_1) does not contain all of these cases. For examples, let $N = 1$ and $L(t) = |t|^\alpha \begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix}$,

we have the eigenvalues of $L(t)$ are $\pm|t|^\alpha$, but there is no constant matrix P satisfying $PL(t) - c|t|^\alpha I_{2N} \geq 0$, $\forall |t| \geq t_0$.

Denote by \tilde{F} the self-adjoint operator $-\mathcal{J}\frac{d}{dt} + L(t)$ on $L^2 \equiv L^2(\mathbb{R}, \mathbb{R}^{2N})$, with domain $D(\tilde{F}) = H^1(\mathbb{R}, \mathbb{R}^{2N})$ if $L(t)$ is bounded and $D(\tilde{F}) \subset H^1(\mathbb{R}, \mathbb{R}^{2N})$ if $L(t)$ is unbounded. Let $|\tilde{F}|$ be the absolute value of \tilde{F} , and $|\tilde{F}|^{1/2}$ be the square root of $|\tilde{F}|$. $D(\tilde{F})$ is a Hilbert space equipped with the norm

$$\|z\|_{\tilde{F}} = \|(I + |\tilde{F}|)z\|_{L^2}, \quad \forall z \in D(\tilde{F}). \quad (1.2)$$

Let $E = D(|\tilde{F}|^{1/2})$, and define on E the inner product and norm by

$$(u, v)_E = (|\tilde{F}|^{1/2}u, |\tilde{F}|^{1/2}v)_2 + (u, v)_2,$$

$$\|u\|_E = (u, u)_E^{1/2},$$

where $(\cdot, \cdot)_{L^2}$ denotes the usual inner product on $L^2(\mathbb{R}, \mathbb{R}^{2N})$. Then E is a Hilbert space. It is easy to see that E is continuously embedded in $H^{1/2}(\mathbb{R}, \mathbb{R}^{2N})$, and we further have the following lemma.

Lemma 1.1. *Suppose that L satisfies (L_1) . Then E is compactly embedded in $L^p(\mathbb{R}, \mathbb{R}^{2N})$ with the usual norm $\|\cdot\|_{L^p}$ for any $1 \leq p \in (\frac{2}{1+\alpha}, \infty)$.*

This lemma is similar to Lemma 2.1-2.3 in [14], and we will prove it in Section 3. Define the quadratic form \mathcal{Q} on E by

$$\mathcal{Q}(u, v) = \int_{\mathbb{R}} ((-Ju, v) + (L(t)u, v))dt, \quad \forall u, v \in E. \quad (1.3)$$

It's easy to check that $\mathcal{Q}(u, v)$ is a bounded quadratic form on E and hence there exists a unique bounded self-adjoint operator $F : E \rightarrow E$ such that

$$(Fu, v)_E = \mathcal{Q}(u, v), \quad \forall u, v \in E. \quad (1.4)$$

Besides, define a linear operator $K : L^2 \rightarrow E$ by

$$(Ku, v)_E = (u, v)_{L^2}, \quad \forall u \in L^2, v \in E. \quad (1.5)$$

In view of Lemma 1.1, we know that F is a Fredholm operator and K is a compact operator.

Denote by \mathcal{B} the set of all uniformly bounded symmetric $2N \times 2N$ matrix functions. That is to say $B \in \mathcal{B}$ if and only if $B^T(t) = B(t)$ for all $t \in \mathbb{R}$ and $B(t)$ is uniformly

bounded in t as the operator on \mathbb{R}^{2N} . For any $B \in \mathcal{B}$, it is easy to see B determines a bounded self-adjoint operator on L^2 , by $z(t) \mapsto B(t)z(t)$, for any $z \in L^2$, we still denote this operator by B , then $KB : E \subset L^2 \rightarrow E$ is a self-adjoint compact operator on E and satisfies

$$(KBu, v)_E = (Bu, v)_{L^2}, \forall u, v \in E. \quad (1.6)$$

Before presenting the conditions on $R(t, z)$, we need the concept of Maslov index for homoclinic orbits introduced by Chen and Hu in [10] which is equivalent to the relative Morse index. We will give a brief introduction of it by Definition 2.1, where for any $B \in \mathcal{B}$, we denote the associated index pair by $(\mu_F(KB), \nu_F(KB))$.

Now we can present the conditions on $R(t, z)$ as follows. For notational simplicity, we set $B_0(t) = \nabla_z^2 R(t, 0)$, and in what follows the letter c will be repeatedly used to denote various positive constants whose exact value is irrelevant. Besides, for two $2N \times 2N$ symmetric matrices M_1 and M_2 , $M_1 \leq M_2$ means that $M_2 - M_1$ is semi-positive definite.

(R_1) $R \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$, and there exists a constant $c > 0$ such that

$$|\nabla_z^2 R(t, z)| \leq c, \quad \forall (t, z) \in \mathbb{R} \times \mathbb{R}^{2N}.$$

(R_0) $\nabla_z R(t, 0) \equiv 0$ and $B_0 \in \mathcal{B}$.

(R_∞) There exists some $R_0 > 0$ and continuous symmetric matrix functions $B_1, B_2 \in \mathcal{B}$ with $\mu_F(KB_1) = \mu_F(KB_2)$ and $\nu_F(KB_2) = 0$ such that

$$B_1(t) \leq \nabla_z^2 R(t, z) \leq B_2(t), \quad \forall t \in \mathbb{R}, |z| > R_0.$$

Then we have our first result.

Theorem 1.2. *Assume (L_1) , (R_1) , (R_0) and (R_∞) hold. If*

$$\mu_F(KB_1) \notin [\mu_F(KB_0), \mu_F(KB_0) + \nu_F(KB_0)],$$

then (HS) has at least one nontrivial homoclinic orbit. Moreover, if $\nu_F(KB_0) = 0$ and $|\mu_F(KB_1) - \mu_F(KB_0)| \geq N$, the problem possesses at least two nontrivial homoclinic orbits.

Condition (R_∞) is a two side pinching condition near the infinity, we can relax (R_∞) to condition (R_∞^\pm) as follows.

(R_∞^\pm) There exists some $R_0 > 0$ and a continuous symmetric matrix function $B_\infty \in \mathcal{B}$ with $v_F(KB_\infty) = 0$ such that

$$\pm \nabla_z^2 R(t, z) \geq \pm B_\infty(t), \quad \forall t \in \mathbb{R}, |z| > R_0.$$

Then we have the following results.

Theorem 1.3. *Assume (L_1) , (R_1) , (R_0) , (R_∞^+) (or (R_∞^-)) and $v_F(KB_0) = 0$ hold. If $\mu_F(KB_\infty) \geq \mu_F(KB_0) + 2$ (or $\mu_F(KB_\infty) \leq \mu_F(KB_0) - 2$), then (HS) has at least one nontrivial homoclinic orbit.*

Theorem 1.4. *Suppose that (L_1) , (R_1) , (R_0) , (R_∞^+) (or (R_∞^-)) and $v_F(KB_0) = 0$ are satisfied. If in addition, R is even in z and $\mu_F(KB_\infty) \geq \mu_F(KB_0) + 2$ (or $\mu_F(KB_\infty) \leq \mu_F(KB_0) - 2$), then (HS) has at least $|\mu_F(KB_\infty) - \mu_F(KB_0)| - 1$ pairs of nontrivial homoclinic orbits.*

Remark 1.5. Lemma 1.1 shows that $\sigma(A)$, the spectrum of A , consists of eigenvalues numbered by (counted in their multiplicities):

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} \leq 0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

with $\lambda_{\pm k} \rightarrow \pm\infty$ as $k \rightarrow \infty$. Let $B_0(t) \equiv B_0$ and $B_\infty(t) \equiv B_\infty$, with the constants B_0, B_∞ satisfying $\lambda_l < B_0 < \lambda_{l+1}$, and $\lambda_{l+i} < B_\infty < \lambda_{l+i+1}$ for some $l \in \mathbb{Z}$ and $i \geq 1$ (or $i \leq -1$). Define

$$R(t, z) = \delta(|z|) \frac{1}{2} B_0 |z|^2 + (1 - \delta(|z|)) \frac{1}{2} B_\infty |z|^2,$$

where δ is a smooth cutoff function satisfying $\delta(|z|) = \begin{cases} 1, & |z| < 1, \\ 0, & |z| > 2. \end{cases}$ By Proposition 2.6 below, it is easy to verify R satisfies all the conditions in Theorem 1.2. Furthermore, let the constant B_∞ satisfying $\lambda_{l+i} < B_\infty < \lambda_{l+i+1}$ for some $l \in \mathbb{Z}$ and $i \geq 2$ (or $i \leq -2$). Define

$$R(t, z) = \delta(|z|) \frac{1}{2} B_0 |z|^2 + (1 - \delta(|z|)) \frac{1}{2} B_\infty |z|^2,$$

Then R satisfies all the conditions in Theorem 1.3 and Theorem 1.4. However, it is easy to see that some conditions of the main results in [14–16, 21] does not hold for these examples.

Remark 1.6. Note that the assumption $v_F(KB_\infty) = 0$ in (R_∞^\pm) is not essential for our main results. For the case of (R_∞^+) with $v_F(KB_\infty) \neq 0$, let $\tilde{B}_\infty = B_\infty - \varepsilon I_{2N}$ with $\varepsilon > 0$

small enough, where I_{2N} is the identity map on \mathbb{R}^{2N} , then $\mu_F(K\tilde{B}_\infty) = \mu_F(KB_\infty)$ and $v_F(K\tilde{B}_\infty) = 0$, and hence (R_∞^+) holds for \tilde{B}_∞ . Therefore Theorems 1.3 and 1.4 still hold in this case. While for the case of (R_∞^-) with $v_F(KB_\infty) \neq 0$, if we replace $\mu_F(KB_\infty)$ by $\mu_F(KB_\infty) + v_F(KB_\infty)$ in Theorems 1.3 and 1.4, then similar results hold. Indeed, let $\tilde{B}_\infty = B_\infty + \varepsilon I_{2N}$ with $\varepsilon > 0$ small enough such that $\mu_F(K\tilde{B}_\infty) = \mu_F(KB_\infty) + v_F(KB_\infty)$ and $v_F(K\tilde{B}_\infty) = 0$, then this case is also reduced to the case of (W_∞^-) for \tilde{B}_∞ with $v_F(K\tilde{B}_\infty) = 0$.

2 Preliminaries

In this section, we recall the definition of relative Morse index, saddle point reduction, and give the relationship between them. For this propose, the notion of spectral flow will be used.

2.1 Relative Morse index

Let \mathcal{H} be a separable Hilbert space, for any self-adjoint operator A on \mathcal{H} , there is a unique A -invariant orthogonal splitting

$$\mathcal{H} = \mathcal{H}^+(A) \oplus \mathcal{H}^-(A) \oplus \mathcal{H}^0(A), \quad (2.1)$$

where $\mathcal{H}^0(A)$ is the null space of A , A is positive definite on $\mathcal{H}^+(A)$ and negative definite on $\mathcal{H}^-(A)$, and P_A denotes the orthogonal projection from \mathcal{H} to $\mathcal{H}^-(A)$. For any bounded self-adjoint Fredholm operator \mathcal{F} and a compact self-adjoint operator \mathcal{T} on \mathcal{H} , $P_{\mathcal{F}} - P_{\mathcal{F}-\mathcal{T}}$ is compact (see Lemma 2.7 of [31]), where $P_{\mathcal{F}} : \mathcal{H} \rightarrow \mathcal{H}^-(\mathcal{F})$ and $P_{\mathcal{F}-\mathcal{T}} : \mathcal{H} \rightarrow \mathcal{H}^-(\mathcal{F} - \mathcal{T})$ are the respective projections. Then by Fredholm operator theory, $P_{\mathcal{F}}|_{\mathcal{H}^-(\mathcal{F}-\mathcal{T})} : \mathcal{H}^-(\mathcal{F} - \mathcal{T}) \rightarrow \mathcal{H}^-(\mathcal{F})$ is a Fredholm operator. Here and in the sequel, we denote by $\text{ind}(\cdot)$ the Fredholm index of a Fredholm operator.

Definition 2.1. For any bounded self-adjoint Fredholm operator \mathcal{F} and a compact self-adjoint operator \mathcal{T} on \mathcal{H} , the relative Morse index pair $(\mu_{\mathcal{F}}(\mathcal{T}), v_{\mathcal{F}}(\mathcal{T}))$ is defined by

$$\mu_{\mathcal{F}}(\mathcal{T}) = \text{ind}(P_{\mathcal{F}}|_{\mathcal{H}^-(\mathcal{F}-\mathcal{T})}). \quad (2.2)$$

and

$$v_{\mathcal{F}}(\mathcal{T}) = \dim \mathcal{H}^0(\mathcal{F} - \mathcal{T}). \quad (2.3)$$

2.2 Saddle point reduction

In this subsection, we describe the saddle point reduction in [4, 8, 24]. Recall that \mathcal{H} is a real Hilbert space, and A is a self-adjoint operator with domain $D(A) \subset \mathcal{H}$. Let $\Phi \in C^1(\mathcal{H}, \mathbb{R})$, with $\Phi'(\theta) = 0$. Assume that

- (1) There exist real numbers $\alpha < \beta$ such that $\alpha, \beta \notin \sigma(A)$, and that $\sigma(A) \cap [\alpha, \beta]$ consists of at most finitely many eigenvalues of finite multiplicities.
- (2) Φ' is Gateaux differentiable in \mathcal{H} , which satisfies

$$\|d\Phi'(u) - \frac{\alpha + \beta}{2}I\| \leq \frac{\beta - \alpha}{2}, \quad \forall u \in \mathcal{H}.$$

Without loss of generality, we may assume $\alpha = -\beta, \beta > 0$.

- (3) $\Phi \in C^2(V, \mathbb{R})$, $V = D(|A|^{1/2})$, with the norm

$$\|z\|_V = (\| |A|^{1/2} z \|_{\mathcal{H}}^2 + \varepsilon^2 \|z\|_{\mathcal{H}}^2)^{1/2},$$

where $\varepsilon > 0$ small and $-\varepsilon \notin \sigma(A)$.

Consider the solutions of the following equation

$$Az = \Phi'(z), \quad z \in D(A). \tag{2.4}$$

Let

$$P_0 = \int_{-\beta}^{\beta} dE_{\lambda}, \quad P_+ = \int_{\beta}^{+\infty} dE_{\lambda}, \quad P_- = \int_{-\infty}^{-\beta} dE_{\lambda},$$

where $\{E_{\lambda}\}$ is the spectral resolution of A , and let

$$\mathcal{H}_* = P_* \mathcal{H}, \quad * = 0, \pm.$$

Decompose the space V as follows

$$V = V_0 \oplus V_- \oplus V_+,$$

where $V_* = |A_{\varepsilon}|^{-1/2} \mathcal{H}_*$, $* = 0, \pm$ and $A_{\varepsilon} = A + \varepsilon I$.

For each $u \in \mathcal{H}$, we have the decomposition

$$u = u_+ + u_0 + u_-,$$

where $u_* \in \mathcal{H}_*$, $* = 0, \pm$, let $z = z_+ + z_0 + z_-$, with

$$z_* = |A_{\varepsilon}|^{-1/2} u_*, \quad * = 0, \pm.$$

Define a functional f on \mathcal{H} as follows:

$$f(u) = \frac{1}{2}(\|u_+\|^2 + Q_+\|u_0\|^2 - Q_-\|u_0\|^2 - \|u\|^2) - \Phi_\varepsilon(z),$$

where $Q_+ = \int_0^\infty dE_\lambda$, $Q_- = \int_{-\infty}^0 dE_\lambda$, and $\Phi_\varepsilon(z) = \frac{\varepsilon}{2}\|z\|_{\mathcal{H}} + \Phi(z)$.

The Euler equation of this functional is the system

$$u_\pm = \pm|A_\varepsilon|^{-1/2}P_\pm\Phi'_\varepsilon(z), \quad (2.5)$$

$$Q_\pm u_0 = \pm|A_\varepsilon|^{-1/2}Q_\pm P_0\Phi'_\varepsilon(z). \quad (2.6)$$

Thus $z = z_+ + z_0 + z_-$ is a solution of (2.4) if and only if $u = u_+ + u_0 + u_-$ is a critical point of f . The implicit function can be applied, yielding a solution $z_\pm(z_0)$ for fixed $z_0 \in V_0$, such that $z_\pm \in C^1(V_0, V_\pm)$. Since $\dim V_0$ is finite, all topologies on V_0 are equivalent, we choose $\|\cdot\|_{\mathcal{H}}$ as it norm. We have

$$u_\pm(z_0) = |A_\varepsilon|^{1/2}z_\pm(z_0) \in C^1(\mathcal{H}_0, \mathcal{H}),$$

which solves the system (2.5).

Let

$$a(z_0) = f(u_+(z_0) + u_-(z_0) + u_0(z_0)),$$

where $u_0(z_0) = |A_\varepsilon|^{1/2}z_0$ and let $z_0 = x$, we have

$$a(x) = \frac{1}{2}(A(z(x), z(x))) - \Phi(z(x)),$$

where $z(x) = \xi(x) + x$, $\xi(x) = z_+(x) + z_-(x) \in D(A)$. Then, we have the following theorem duo to Amann and Zehnder [4], Chang [8] and Long [24].

Theorem 2.2. *Under the assumption (1), (2), (3), there is a one-one correspondence*

$$x \mapsto z = z(x) = z_+(x) + z_-(x) + x,$$

between the critical points of the C^2 -function $a \in C^2(\mathcal{H}_0, \mathbb{R})$ with the solutions of the operator equation

$$Az = \Phi'(z), \quad z \in D(A).$$

Moreover, the functional a satisfies

$$a(x) = \frac{1}{2}(A(z(x), z(x))) - \Phi(z(x)),$$

$$a'(x) = A(z(x)) - \Phi'(z(x)) = Ax - P_0\Phi'(z(x)),$$

$$a''(x) = [A - \Phi''(z(x))]z'(x) = AP_0 - P_0\Phi''(z(x))z'(x).$$

Since \mathcal{H}_0 is a finite dimensional space, for every critical point x of a in \mathcal{H}_0 , the Morse index and nullity are finite, we denote them by $(m_a^-(x), m_a^0(x))$.

Now, let the Hilbert space \mathcal{H} be $L^2(\mathbb{R}, \mathbb{R}^{2N})$, and the operator A be $\tilde{F} = -J \frac{d}{dt} + L$, $\Phi(u) = \int_{-\infty}^{\infty} R(t, u)$. Then we have $V = E$. For $R \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ and $|\nabla_z^2 R| \leq C_R$, $\forall (t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$, let $-\alpha = \beta \geq 2(C_R + 1)$ and $\beta \notin \sigma(\tilde{F})$, we have A and Φ satisfying the above conditions. Thus from Theorem 2.2, we can solve our problems on the finite dimensional space. Similar to Lemma 2.2 and Remark 2.3 in [23], we have the following estimates.

Lemma 2.3. *Assume that $R \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$, $|\nabla_z^2 R| \leq C_R$, $\forall (t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$ and $\nabla_z R(t, 0) \equiv 0$, then we have*

$$\|u^\pm(x)\|_{L^2} \leq \frac{2\sqrt{\beta}(C_R + 1)}{\beta - 2C_R - 3\varepsilon} \|x\|_{L^2}, \forall x \in \mathcal{H}_0.$$

Moreover, we have

$$\|(u^\pm)'(x)\|_{L^2} \rightarrow 0, \beta \rightarrow \infty.$$

Proof. Note that

$$u^\pm(x) = \pm |A_\varepsilon|^{-1/2} P_\pm \Phi'_\varepsilon(z^+ + z^- + x).$$

From $\nabla_z R(t, 0) = 0$, $|\nabla_z^2 R| \leq C_R$, we have $\Phi'(0) = 0$ and $\|\Phi'(z)\|_{L^2} \leq C_R \|z\|_{L^2}$. Since $\||A_\varepsilon|^{-1/2} P_\pm\| \leq \frac{1}{\sqrt{\beta - \varepsilon}}$, we have

$$\begin{aligned} \|u^\pm(x)\|_{L^2} &\leq \frac{1}{\sqrt{\beta - \varepsilon}} \|\Phi'(z^+ + z^- + x) + \varepsilon(z^+ + z^- + x)\|_{L^2} \\ &\leq \frac{C_R + \varepsilon}{\sqrt{\beta - \varepsilon}} \|z^+ + z^- + x\|_{L^2} \\ &\leq \frac{C_R + \varepsilon}{\sqrt{\beta - \varepsilon}} \left(\frac{\|u^+(x)\|_{L^2}}{\sqrt{\beta}} + \frac{\|u^-(x)\|_{L^2}}{\sqrt{\beta}} + \|x\|_{L^2} \right). \end{aligned} \quad (2.7)$$

Therefore,

$$\|u^+(x)\|_{L^2} + \|u^-(x)\|_{L^2} \leq \frac{2\sqrt{\beta}(C_R + \varepsilon)}{\beta - 2C_R - 3\varepsilon} \|x\|_{L^2}.$$

Next, since

$$(u^\pm)'(x) = \pm |A_\varepsilon|^{-1/2} P_\pm \Phi'_\varepsilon(z^+ + z^- + x)((z^+)'(x) + (z^-)'(x) + I),$$

where I is the identity map on \mathcal{H}_0 , we have

$$\|(u^+)'(x)\|_{L^2} + \|(u^-)'(x)\|_{L^2} \leq \frac{2\sqrt{\beta}(C_R + \varepsilon)}{\beta - 2C_R - 3\varepsilon}, \forall x \in \mathcal{H}_0.$$

Remark 2.4. For $z(x)$, we also have that there is a constant $C > 0$ dependent of C_R , but independent of β , such that

$$\|z^\pm(x)\|_V \leq \frac{C}{\sqrt{\beta}} \|x\|_{L^2}, \|z'^\pm(x)\|_V \leq \frac{C}{\sqrt{\beta}}, \forall x \in \mathcal{H}_0.$$

If R satisfies the condition (R_1) , then for any homoclinic orbit z of (HS), $\nabla_z^2 R(\cdot, z) \in \mathcal{B}$, and hence we have the associated index pair $(\mu_F(KB), \nu_F(KB))$. For notation simplicity, in what follows, we set

$$\mu_F(z) = \mu_F(K\nabla_z^2(R(t, z))),$$

and

$$\nu_F(z) = \nu_F(K\nabla_z^2(R(t, z))).$$

Theorem 2.5. *Let $R \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ satisfying $|\nabla_z^2 R| \leq C_R$, $\forall (t, z) \in \mathbb{R} \times \mathbb{R}^{2N}$ and $\nabla_z R(t, 0) \equiv 0$. For each critical point x of a in \mathcal{H}_0 , $z(x)$ is a homoclinic orbit of (HS) and we have*

$$m_a^-(x) = \dim(E^-(\mathcal{H}_0)) + \mu_F(K\nabla_z^2 R(t, z(x))) = \dim(E^-(\mathcal{H}_0)) + \mu_F(z(x)), \quad (2.8)$$

$$m_a^0(x) = \nu_F(K\nabla_z^2 R(t, z(x))) = \nu_F(z(x)), \quad (2.9)$$

where $\dim(E^-(\mathcal{H}_0))$ is the dimension of the space $\int_{-\beta}^{0^-} dE_\lambda(\mathcal{H}_0)$.

This theorem shows the relations between the relative Morse index and the Morse index of the saddle point reduction, it will play an important role in the proof of our main results. The proof of this theorem will be postponed in the next subsection where the notion of spectral flow will be used.

2.3 The relationship between $\mu_F(T)$, spectral flow and the Morse index of saddle point reduction

It is well known that the concept of spectral flow was first introduced by Atiyah, Patodi and Singer in [6], and then extensively studied in [7, 18, 25, 26, 31]. Here, we give a brief introduction of the spectral flow as introduced in [10]. Let \mathcal{H} be a separable Hilbert space as defined before, and $\{\mathcal{F}_\theta | \theta \in [0, 1]\}$ be a continuous path of self-adjoint Fredholm operators on the Hilbert space \mathcal{H} . The spectral flow of \mathcal{F}_θ represents the net change in the number of negative eigenvalues of \mathcal{F}_θ as θ runs from 0 to 1, where the counting follows from the rule that each negative eigenvalue crossing to the positive axis contributes +1 and each positive eigenvalues crossing to the negative axis contributes -1, and for each

crossing the multiplicity of eigenvalue is taken into account. In the calculation of spectral flow, a crossing operator introduced in [26] will be used. Take a C^1 path $\{\mathcal{F}_\theta | \theta \in [0, 1]\}$ and let \mathcal{P}_θ be the projection from \mathcal{H} to $\mathcal{H}^0(\mathcal{F}_\theta)$. When eigenvalue crossing occurs at \mathcal{F}_θ , the operator

$$\mathcal{P}_\theta \frac{\partial}{\partial \theta} \mathcal{F}_\theta \mathcal{P}_\theta : \mathcal{H}^0(\mathcal{F}_\theta) \rightarrow \mathcal{H}^0(\mathcal{F}_\theta) \quad (2.10)$$

is called a crossing operator, denoted by $C_r[\mathcal{F}_\theta]$. As mentioned in [26], an eigenvalue crossing at \mathcal{F}_θ is said to be regular if the null space of $C_r[\mathcal{F}_\theta]$ is trivial. In this case, we define

$$\text{sign } C_r[\mathcal{F}_\theta] = \dim \mathcal{H}^+(C_r[\mathcal{F}_\theta]) - \dim \mathcal{H}^-(C_r[\mathcal{F}_\theta]). \quad (2.11)$$

A crossing occurs at \mathcal{F}_θ is called simple crossing if $\dim \mathcal{H}^0(\mathcal{F}_\theta) = 1$.

As indicated in [31], the spectral flow $Sf(\mathcal{F}_\theta)$ will remain the same after a small disturbance of \mathcal{F}_θ , that is, $Sf(\mathcal{F}_\theta) = Sf(\mathcal{F}_\theta + \varepsilon id)$ for $\varepsilon > 0$ and small enough, where id is the identity map on \mathcal{H} . Furthermore, we can choose suitable ε such that all the eigenvalue crossings occurred in $\mathcal{F}_\theta, 0 \leq \theta \leq 1$ are regular [26]. Thus, without loss of generality, we may assume all the crossings are regular. Let \mathcal{D} be the set containing all the points in $[0, 1]$ at which the crossing occurs. The set \mathcal{D} contains only finitely many points. The spectral flow of \mathcal{F}_θ is

$$Sf(\mathcal{F}_\theta, 0 \leq \theta \leq 1) = \sum_{\theta \in \mathcal{D}^*} \text{sign } C_r[\mathcal{F}_\theta] - \dim \mathcal{H}^-(C_r[\mathcal{F}_0]) + \dim \mathcal{H}^+(C_r[\mathcal{F}_1]), \quad (2.12)$$

where $\mathcal{D}^* = \mathcal{D} \cap (0, 1)$. In what follows, the spectral flow of \mathcal{F}_θ will be simply denoted by $Sf(\mathcal{F}_\theta)$ when the starting and end points of the flow are clear from the contents. And $P_{\mathcal{F}_\theta}$ will be simply denoted by P_θ .

Proposition 2.6. (See [10, Proposition 3].) Suppose that, for each $\theta \in [0, 1]$, $\mathcal{F}_\theta - \mathcal{F}_0$ is a compact operator on \mathcal{H} , then

$$\text{ind}(P_0|_{\mathcal{H}^-(\mathcal{F}_1)}) = -Sf(\mathcal{F}_\theta).$$

Thus, from Definition 2.1,

$$\mu_{\mathcal{F}_0}(\mathcal{T}) = -Sf(\mathcal{F}_\theta, 0 \leq \theta \leq 1),$$

where $\mathcal{F}_\theta = \mathcal{F} - \theta \mathcal{T}$, \mathcal{T} is a compact operator. More over, if $\sigma(\mathcal{T}) \subset [0, \infty)$ and $0 \notin \sigma_P(\mathcal{T})$,

from the definition of Spectral flow, we have

$$\begin{aligned}
\mu_{\mathcal{F}_0}(\mathcal{T}) &= -Sf(\mathcal{F}_\theta, 0 \leq \theta \leq 1) \\
&= \sum_{\theta \in [0,1)} v_{\mathcal{F}}(\theta \mathcal{T}) \\
&= \sum_{\theta \in [0,1)} \dim \mathcal{H}^0(\mathcal{F} - \theta \mathcal{T}).
\end{aligned} \tag{2.13}$$

The proof of Theorem 2.5 is the direct consequence of the above Proposition 2.6 and Theorem 3.2 in [31], so we omit it here.

Remark 2.7. *The case of R_∞^- can be transformed into the case of R_∞^+ . More concretely, the R_∞^- case follows from the R_∞^+ case by applying to the function $\tilde{R}(t, z) = -R(-t, z)$. If $z(t)$ is a homoclinic solution of $\tilde{F}z(t) = \nabla_z R(t, z(t))$, let $\tilde{z}(t) = z(-t)$, it's easy to check that $\tilde{z}(t)$ is a homoclinic solution of $\tilde{F}\tilde{z}(t) = \nabla_z \tilde{R}(t, \tilde{z}(t))$, and this is a one-one correspondence between the two systems. By the definition of spectral flow and its catenation property [31], we have $\mu_F(-B_\infty(-t)) - \mu_F(-B_0(-t)) = \mu_F(B_0(t)) - \mu_F(B_\infty(t))$. Thus, we only consider the case of R_∞^+ from now on.*

3 Proof of our main results

Proof of Lemma 1.1. Recall the operator $\tilde{F} = -\mathcal{J} \frac{d}{dt} + L(t)$, with domain $D(\tilde{F}) = H^1(\mathbb{R}, \mathbb{R}^{2N})$ if $L(t)$ is bounded and $D(\tilde{F}) \subset H^1(\mathbb{R}, \mathbb{R}^{2N})$ if $L(t)$ is unbounded. $D(\tilde{F})$ is a Hilbert space equipped with the norm $\|z\|_{\tilde{F}} = \|(I + |\tilde{F}|)z\|_{L^2}$, $\forall z \in D(\tilde{F})$. Recall the Hilbert space $E = D(|\tilde{F}|^{1/2})$, with the inner product and norm by

$$(u, v)_E = (|\tilde{F}|^{1/2}u, |\tilde{F}|^{1/2}v)_2 + (u, v)_2,$$

$$\|u\|_E = (u, u)_E^{1/2},$$

where $(\cdot, \cdot)_{L^2}$ denotes the usual inner product on $L^2(\mathbb{R}, \mathbb{R}^{2N})$. From (L_1) , there is a matrix L_0 such that $P(L(t) - L_0) \geq 0$ for all $t \in \mathbb{R}$. We have $L(t) = \tilde{F} - \tilde{F}_0 + L_0$, with $\tilde{F}_0 = -\mathcal{J} \frac{d}{dt} + L_0$ and $D(\tilde{F}_0) = H^1$. Thus, for any $z \in E$

$$\begin{aligned}
|(L(t)z, P^T z)_{L^2}| &\leq |(\tilde{F}z, P^T z)_{L^2}| + |(\tilde{F}_0 z, P^T z)_{L^2}| + |(L_0 z, P^T z)_{L^2}| \\
&\leq c\|z\|_E^2.
\end{aligned} \tag{3.1}$$

Let $K \subset E$ be a bounded set. We will show that K is precompact in L^p for $1 \leq p \in (2/(1 + \alpha), \infty)$. We derive the proof into three steps.

Step 1. The case of $p = 2$. For $R > 0$, from (L_1) and (3.1) we have

$$\begin{aligned} \int_{|t|>R} |z|^2 &\leq c|R|^{\alpha-2} \int_{|t|>R} \langle L(t)z, P^T z \rangle_{\mathbb{R}^{2N}} \\ &\leq c|R|^{\alpha-2} \|z\|_E^2. \end{aligned} \quad (3.2)$$

For any $\varepsilon > 0$, from (3.2), we can choose R_0 large enough, such that

$$\int_{|t|>R_0} |z|^2 < \frac{\varepsilon^2}{4}, \forall z \in K. \quad (3.3)$$

On the other hand, by the definition of $\|\cdot\|_E$, we have

$$\int_{|t|\leq R_0} |z|^2 \leq \|z\|_E^2 \leq C, \forall z \in K. \quad (3.4)$$

Thus, by the Sobolev compact embedding theorem there exist $z_1, z_2, \dots, z_m \in K$, such that for any $z \in K$ there is z_i satisfying

$$\|z - z_i\|_{L^p((-R_0, R_0), \mathbb{R}^{2N})}^p < \frac{\varepsilon^2}{2}. \quad (3.5)$$

From (3.3) and (3.5), we have $\|z - z_i\|_{L^2} < \varepsilon$, thus, K has a finite ε -net in L^2 , so the embedding $E \hookrightarrow L^2$ is compact.

Step 2. The case of $p > 2$. Since E is continuously embedded in $H^{1/2}$, hence by the Sobolev embedding theorem, E is continuously embedded in L^p , $\forall p > 2$. For any $p > 2$, by the Hölder inequality we have

$$\int_{\mathbb{R}} |z|^p \leq \|z\|_{L^2} \|z\|_{L^{2(p-1)}}^{p-1} \leq C \|z\|_{L^2} \|z\|_E^{p-1},$$

thus, the embedding $E \hookrightarrow L^p$ is compact, $\forall p > 2$.

Step 3. The case of $1 \leq p \in (2/(1+\alpha), 2)$. First, we have $\frac{\alpha}{2-p} \cdot p > 1$, so we can choose α_p satisfying $\alpha_p \in (0, \alpha)$ and $\frac{\alpha_p}{2-p} \cdot p > 1$. Denote by $r = \frac{\alpha_p}{2-p}$. For $R > 0$ and $z \in E$, denote by $E_R^1(z) = \{t; |t| \geq R \text{ and } |t|^r |z(t)| > 1\}$ and $E_R^2(z) = \{t; |t| \geq R \text{ and } |t|^r |z(t)| \leq 1\}$. Then, from (3.1),

$$\begin{aligned} \int_{E_R^1(z)} |z|^p &= \int_{E_R^1(z)} (|t|^r |z|)^p |t|^{-rp} \\ &\leq \int_{E_R^1(z)} |z|^2 |t|^{\alpha_p} \\ &\leq \frac{c}{|R|^{\alpha-\alpha_p}} |(L(t)z, P^T z)_{L^2}| \\ &\leq \frac{c}{|R|^{\alpha-\alpha_p}} \|z\|_E^2, \end{aligned} \quad (3.6)$$

and so

$$\begin{aligned} \int_{|t| \geq R} |z|^p &= \int_{E_R^1(z)} |z|^p + \int_{E_R^2(z)} |z|^p \\ &\leq \frac{c}{|R|^{\alpha-\alpha_p}} \|z\|_E^2 + \frac{2}{(rp-1)R^{rp-1}}, \quad \forall z \in E. \end{aligned} \quad (3.7)$$

Let $K \subset E$ be a bounded set. For any $\varepsilon > 0$, from (3.7), choose $R_0 > 0$ large enough, such that

$$\int_{|t| \geq R_0} |z|^p < \frac{\varepsilon^p}{4}, \quad \forall z \in K. \quad (3.8)$$

On the other hand, by the Sobolev compact embedding theorem there are $z_1, z_2, \dots, z_m \in K$, such that for any $z \in K$, there exists z_i satisfying

$$\|z - z_i\|_{L^p((-R_0, R_0), \mathbb{R}^{2N})}^p < \frac{\varepsilon^p}{2} \quad (3.9)$$

From (3.8) and (3.9), we have

$$\|z - z_i\|_{L^p} < \varepsilon,$$

that is to say K has a finite ε -net in L^p , and the embedding $E \hookrightarrow L^p$ is compact. The proof of the lemma is compact. \square

Consider the homoclinic orbits of the linear Hamiltonian systems

$$\begin{cases} \dot{z}(t) = JB(t)z(t), \forall t \in \mathbb{R}, \\ z(t) \rightarrow 0, |t| \rightarrow \infty. \end{cases} \quad (3.10)$$

where $z(t) : \mathbb{R} \rightarrow \mathbb{R}^{2N}$, $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ and $B(t)$ is a continuous symmetric matrix function. Denote by S the set of homoclinic orbits of linear systems (3.10), then S is a linear subspace of $L^2(\mathbb{R}, \mathbb{R}^{2N})$ and we have the following lemma.

Lemma 3.1. *The dimension of the solution space S will be less than or equal to N . Thus for any homoclinic orbit $z(t)$ of (HS), if R satisfies (R_1) , we have*

$$0 \leq v_F(z) \leq N.$$

Proof. As usual, we define the symplectic groups on \mathbb{R}^{2N} by

$$Sp(2N) = \{M \in \mathcal{L}(\mathbb{R}^{2N}), |M^T J M = J\},$$

where $\mathcal{L}(\mathbb{R}^{2N})$ is the set of all $2N \times 2N$ real matrices, M^T denotes the transpose of M . Let $W(t)$ be the fundamental solution of (3.10), then $W(t)$ is a path in $Sp(2N)$. Let $z(t)$ be a nontrivial homoclinic orbits of (3.10), that is to say $z(0) \neq 0$ and satisfies

$$\begin{cases} z(t) = W(t)z(0), \\ \lim_{t \rightarrow \infty} W(t)z(0) = 0. \end{cases}$$

Denote by S_0 the subset of \mathbb{R}^{2N} satisfying

$$S_0 = \{z \in \mathbb{R}^{2N} \mid \lim_{t \rightarrow \infty} W(t)z = 0\},$$

then we have $\dim S = \dim(S_0)$. We claim that $Jz_0 \notin S_0$ if $z_0 \in S_0$ and $z_0 \neq 0$. We prove it indirectly, assume $z_0, Jz_0 \in S_0$ with $z_0 \neq 0$, that is to say

$$\lim_{t \rightarrow \infty} W(t)z_0 = 0,$$

$$\lim_{t \rightarrow \infty} W(t)Jz_0 = 0.$$

Since $W(t)$ is a path in $Sp(2N)$, $W^T(t)JW(t) = J$, $\forall t \in \mathbb{R}$, thus

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} (JW(t)z_0, W(t)Jz_0)_{\mathbb{R}^{2N}} \\ &= - \lim_{t \rightarrow \infty} (z_0, W^T(t)JW(t)Jz_0)_{\mathbb{R}^{2N}} \\ &= (z_0, z_0)_{\mathbb{R}^{2N}}, \end{aligned}$$

which contradicts $z_0 \neq 0$. Since J is an isomorphism on \mathbb{R}^{2N} , we have $\dim S_0 \leq N$. And from the definition of $v_F(z)$ in the last part of subsection 2.2, we have completed the proof. \square

Before the proof of Theorem 1.2, we need the following lemma. Since R satisfies condition (R_1) , performing on (HS) the saddle point reduction. Choose a suitable number β , which is used in the projection for the saddle point reduction in section 2.2. Let

$$P = \int_{-\beta}^{\beta} dE_{\lambda}, \quad (3.11)$$

$$X = PL^2(\mathbb{R}, \mathbb{R}^{2N}). \quad (3.12)$$

By Theorem 2.2, we have a functional $a(x)$ with $x \in X$, whose critical points give rise to solutions of (HS).

Lemma 3.2. (1) a satisfies (PS) condition,

(2) $H_q(X, a; \mathbb{R}) \cong \delta_{q,r} \mathbb{R}$, $q = 0, 1, \dots$ for $-a \in \mathbb{R}$ large enough, where $r = \dim(E^-(X)) + \mu_F(KB_1)$.

Proof. Assume there is a sequence $\{x_n\} \subset X$, satisfying $a'(x_n) \rightarrow 0$ ($n \rightarrow \infty$). That is

$$\|Fz_n - K\nabla_z R(t, z_n)\|_E \rightarrow 0, \quad (3.13)$$

where $z_n = z(x_n)$ defined in section 2.1. Since X is a finite dimensional space, and from the definition of z_n , it's enough to prove $\{z_n\}$ is bounded in E . For each $\varepsilon \in (0, 1)$, define $C_n \in \mathcal{B}$ by

$$C_n(t) = \begin{cases} \int_0^1 \nabla_z^2 R(t, sz_n) ds, & |z_n(t)| \geq \frac{R_0}{\varepsilon}, \\ B_1(t), & |z_n(t)| < \frac{R_0}{\varepsilon}. \end{cases} \quad (3.14)$$

It is easy to verify that $\{C_n\}$ satisfies

$$B_1(t) - \varepsilon(B_1(t) + c \cdot I) \leq C_n(t) \leq B_2(t) + \varepsilon(c \cdot I - B_2(t)), \quad \forall t \in \mathbb{R},$$

where c is the constant in condition (R_1) and I is the identity map on \mathbb{R}^{2N} . Since $B_1 \leq B_2$, $\mu_F(B_1) = \mu_F(B_2)$ and $v_F(KB_1) = v_F(KB_2) = 0$, we can choose ε small enough, such that for each $n \in \mathbb{N}^+$, satisfying $\mu_F(KC_n) = \mu_F(KB_1)$ and $v_F(KC_n) = 0$. Thus $F - KC_n$ is reversible on E and there is a constant $\delta > 0$, such that

$$\|(F - KC_n)z\|_E \geq \delta \|z\|_E, \quad \forall z \in E, n \in \mathbb{N}^+. \quad (3.15)$$

On the other hand, for $b \in (0, 1)$, there is a constant $c > 0$ depending on b , such that for each $n \in \mathbb{N}^+$,

$$|\nabla_z R(t, z_n(t)) - C_n z_n(t)| \leq c |z_n(t)|^b, \quad \forall t \in \mathbb{R}. \quad (3.16)$$

Choose $b > \frac{1-\alpha}{1+\alpha}$ in (3.16), that is $1+b \in (\frac{2}{1+\alpha}, 2)$, we have

$$\begin{aligned} \|(Fz_n - K\nabla_z R(t, z_n)) - (F - KC_n)z_n\|_E^2 &= \|K(\nabla_z R(t, z_n) - C_n z_n)\|_E^2 \\ &\leq \|\nabla_z R(t, z_n) - C_n z_n\|_{L^2}^2 \\ &\leq c \int_{\mathbb{R}} \frac{|\nabla_z R(t, z_n) - C_n z_n|}{|z_n|^b} |z_n|^{1+b} dt \\ &\leq c \|z_n\|_{L^{1+b}}^{1+b}. \end{aligned} \quad (3.17)$$

As we claimed in the part of introduction, in equations (3.16) and (3.17) the letter c denotes different positive constants whose exact value is irrelevant. Thus, from (3.13), (3.15), (3.17) and Lemma 1.1, we have $\{z_n\}$ is bounded in E , and a satisfies the (PS) conditions. And by Lemma 5.1 in Chapter II of [8], we have

$$H_q(X, (a)_\alpha; \mathbb{R}) \cong \delta_{q,r} \mathbb{R}, \quad q = 0, 1, \dots,$$

for $-\alpha \in \mathbb{R}$ large enough. \square

From Theorem 2.5, Lemma 3.1 and Lemma 3.2, Theorem 1.2 is a direct consequence of Theorem 5.1 and Corollary 5.2 in chapter II of [8].

In order to proof Theorem 1.3 and Theorem 1.4, we need the following lemma which is similar to Lemma 3.4 in [22] and Lemma 3.3 in [23].

Lemma 3.3. Assume (R_1) , (R_0) and (R_∞^+) hold, then there exists a sequence of functions $R_k \in C^2(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$, $k \in \mathbb{N}$, satisfying the following properties:

(1) There exists an increasing sequence of real numbers $M_k \rightarrow \infty (k \rightarrow \infty)$ such that

$$R_k(t, z) \equiv R(t, z), \quad \forall t \in \mathbb{R}, |z| \leq M_k; \quad (3.18)$$

(2) For each $k \in \mathbb{N}$, there is a $C > 0$ independent of k , such that

$$|\nabla_z^2 R_k(t, z)| \leq C, \quad \forall t \in \mathbb{R}, z \in \mathbb{R}^{2N}, \quad (3.19)$$

$$\nabla_z^2 R_k(t, z) \geq B_\infty, \quad \forall t \in \mathbb{R}, |z| \geq R_0. \quad (3.20)$$

(3) For each $k \in \mathbb{N}$, there exists some $C_k > 0$ and a constant γ with $\gamma I_{2N} > B_\infty$, $\nu_F(K\gamma I_{2N}) = 0$ such that

$$|\nabla_z R_k(t, z) - \gamma z| < C_k, \quad \forall (t, z) \in \mathbb{R} \times \mathbb{R}^{2N}, \quad (3.21)$$

where I_{2N} is the identity map on \mathbb{R}^{2N} .

Proof. Define $\eta : [0, \infty) \rightarrow \mathbb{R}$ by

$$\eta(s) = \begin{cases} 0, & 0 \leq s < 1, \\ \frac{2}{9}(s-1)^3 - \frac{1}{9}(s-1)^4, & 1 \leq s < 2, \\ 1 - \frac{128}{9(12+s^2)}, & 2 \leq s < \infty. \end{cases}$$

It's easy to see that $\eta \in C^2([0, \infty), \mathbb{R})$. Choose a sequence $\{M_k\}$ of positive numbers such that $R_0 < M_1 < M_2 < \dots < M_k < \dots \rightarrow \infty$ as $k \rightarrow \infty$. For each $k \in \mathbb{N}$, let $\eta_k(s) = \eta(\frac{s}{M_k})$ and

$$R_k(t, z) = (1 - \eta_k(|z|))R(t, z) + \frac{\gamma}{2}\eta_k(|z|)|z|^2, \quad k \in \mathbb{N}. \quad (3.22)$$

As in [22, 23], we can check that R_k satisfies (3.18)–(3.21) for each $k \in \mathbb{N}$. \square

For each $k \in \mathbb{N}$, we consider the following problem

$$\begin{cases} \tilde{F}z = \nabla_z R_k(t, z), \\ z(t) \rightarrow 0, z'(t) \rightarrow 0, t \rightarrow \infty \end{cases} \quad (HS)_k$$

where R_k is given in Lemma 3.3. Performing on $(HS)_k$ the saddle point reduction. We choose the number β which is used in the projection for the saddle point reduction in section 2.2. First we choose

$$\beta > \max\{2(C+1), 2(\gamma+1)\}, \text{ and } \beta \notin \sigma(A_0).$$

Let

$$P_\beta = \int_{-\beta}^{\beta} dE_\lambda, \quad (3.23)$$

$$X_\beta = P_\beta L^2(\mathbb{R}, \mathbb{R}^{2N}). \quad (3.24)$$

Thus for each k and such a β fixed, by Theorem 2.2, we have a functional

$$a_{k,\beta}(x), \quad x \in X_\beta,$$

whose critical points give rise to solutions of $(HS)_k$. Similarly we have a functional

$$a_{\gamma,\beta}(x), \quad x \in X_\beta,$$

whose critical points give rise to solutions of the following systems $(HS)_\gamma$

$$\begin{cases} \tilde{F}z = \gamma z, \\ z(t) \rightarrow 0, z'(t) \rightarrow 0, t \rightarrow \infty. \end{cases} \quad (HS)_\gamma$$

For notational simplicity, we denote a_k, a_γ for $a_{k,\beta}$ and $a_{\gamma,\beta}$. Define

$$\Phi_k(z) = \int_{-\infty}^{\infty} R_k(t, z).$$

For the functional a_k , similar to Lemma 3.2, we have the following lemma.

Lemma 3.4.

- (1) a_k satisfies (PS) condition, the critical point set of a_k is compact,
- (2) $H_q(X_\beta, (a_k)_{\alpha_k}; \mathbb{R}) \cong \delta_{q,r_\beta} \mathbb{R}, q = 0, 1, \dots$ for $-\alpha_k \in \mathbb{R}$ large enough, where $r_\beta = \dim(E^-(X_\beta)) + \mu_F(K\gamma I_{2N})$.

Proof. The proof is similar to Lemma 3.2. From Theorem 2.5, we have

$$\begin{aligned} \|a'_k(x) - a''_\gamma(0)x\|_{L^2} &= \|P_\beta K(\nabla_z \Phi_k(z_k(x)) - \gamma x)\|_{L^2} \\ &\leq \|P_\beta(\nabla_z R_k(z_k(x)) - \gamma z_k(x))\|_{L^2} \\ &\leq \|(\nabla_z R_k(z_k(x)) - \gamma z_k(x))\|_{L^2}. \end{aligned} \quad (3.25)$$

Similar to (3.16), we have for $b \in (0, 1)$, there is some $c > 0$, such that

$$|\nabla_z R_k(t, z_k) - \gamma z_k| \leq c|z_k(t)|, \quad \forall t \in \mathbb{R},$$

Choose $b \in (\frac{1-\alpha}{1+\alpha}, 1)$, similar to (3.17), we have

$$\|\nabla_z R_k(t, z_k) - \gamma z_k\|_{L^2}^2 \leq c\|z_k\|_{L^{1+b}}^{1+b}. \quad (3.26)$$

From Lemma 1.1, Remark 2.4 and equation (3.26),

$$\begin{aligned}\|\nabla_z R_k(t, z_k) - \gamma z_k\|_{L^2}^2 &\leq c\|z_k\|_E^{1+b} \\ &\leq c(\beta)\|x\|_{L^2}^{1+b}.\end{aligned}\tag{3.27}$$

From (3.25) and (3.27), we have

$$\|a'_k(x) - a''_\gamma(0)x\|_{L^2}^2 \leq c(\beta)\|x\|_{L^2}^{1+b}.\tag{3.28}$$

Now, for each $k \in \mathbb{N}$, we assume $\{x_m\} \subset X_\beta$ satisfying $\|a'_k(x_m)\| \rightarrow 0$. By $\nu_F(K\gamma I_{2N}) = 0$, we have $a''_\gamma(0)$ is invertible on X_β , since $b < 1$ the sequence $\{x_m\}$ must be bounded. Thus the (PS) condition for a_k holds. From the same reason, we have the compactness of the critical point set of a_k . And by Lemma 5.1 in Chapter II of [8], we have

$$H_q(X_\beta, (a_k)_{\alpha_k}; \mathbb{R}) \cong \delta_{q, r_\beta} \mathbb{R}, q = 0, 1, \dots,$$

for $-\alpha_k \in \mathbb{R}$ large enough. \square

Lemma 3.5. *There exist $c > 0$, such that for any $k \in \mathbb{N}$, and $z \in L^2$ satisfies the systems $(HS)_k$, if $\mu_F(z) \leq \mu_F(KB_\infty) - 1$, we have $\|z\|_{L^\infty} \leq c$.*

Proof. We prove it indirectly. Assume there exist R_k, z_k , satisfies the conditions, and $\|z_k\|_{L^\infty} \rightarrow \infty$, that is $\|z\|_{\tilde{F}} \rightarrow \infty$. Since $|\nabla_z R_k(t, z)| < c|z|$, $\forall t \in \mathbb{R}, z \in \mathbb{R}^{2N}$, we have $\|z_k\|_{\tilde{F}} \leq c\|z_k\|_{L^2}$. Denote $y_k = \frac{z_k}{\|z_k\|_{\tilde{F}}}$, then we have $y_k \rightarrow y$ in L^2 for some $y \in L^2$ with $\|y\|_{L^2} > 0$, and

$$\tilde{F}y_k = \frac{R'_k(t, z_k)}{\|z_k\|_{\tilde{F}}}.$$

Then for any $r > 0$, there exist $C_r > 0$, satisfying

$$|\dot{y}_k(t)| \leq C_r|y_k(t)|, t \in I_r,\tag{3.29}$$

where $I_r = [-r, r]$. Since $\|y\|_{L^2} > 0$, there is a $r_0 > 0$, such that

$$\|y\|_{L^2(-r, r)} > \frac{1}{2}\|y\|_{L^2} > 0, \forall r > r_0.$$

Then from the similar argument in [23], there is a subsequence we may assume $\{y_k\}$ converges in uniform norm to y , and $y(t) \neq 0, \forall t \in I_r$. Therefor $|z_k(t)| \rightarrow \infty$ uniformly on I_r , and there is $K(r)$ depending on r , such that $|z_k(t)| \geq R_0$, for any $t \in I_r$ and $k \geq K(r)$.

Performing the saddle point reduction on the following systems

$$\begin{cases} \tilde{F}z = B_\infty z, \\ z(t) \rightarrow 0, z'(t) \rightarrow 0, t \rightarrow \infty. \end{cases}\tag{HS}_\infty$$

For β large enough, we have the functional $a_{\infty,\beta}$ (denote by a_∞ for simplicity) and the function $z(x)$, since $v_F(KB_\infty) = 0$, we have the following decomposition

$$X_\beta = X_\beta^+ + X_\beta^-,$$

where $a_\infty''(0)$ is positive definite on X_β^+ and negative definite on X_β^- . From Remark 2.4, and $v_F(KB_\infty) = 0$, there exists $\alpha > 0$, such that for β large enough

$$((\tilde{F} - B_\infty(t))x, x)_{L^2} \leq -\alpha\|x\|_{L^2}^2, \quad \forall x \in X_\beta^-. \quad (3.30)$$

From the uniform boundary of $\nabla_z^2 R_k(t, z)$ and Remark 2.4, we can choose β large enough, such that

$$\|(\nabla_z^2 R_k(t, z_k) z_k'^{\pm}(x_k) x, x)\|_{L^2} \leq \frac{\alpha}{4} \|x\|_{L^2}, \quad \forall x \in L^2, \quad (3.31)$$

where $x_k = P_\beta z_k$, $z_k(x_k) = z_k(t)$ defined in Theorem 2.2. Choose $\varepsilon > 0$ small enough and $\varepsilon < \frac{\alpha}{4}$, such that $\mu_F(KB_\infty) = \mu_F(K(B_\infty - \varepsilon \cdot Id))$. Since X_β^- is finite dimensional space, choose r large enough, such that

$$((\nabla_z^2 R_k(t, z_k) - B_\infty)x, x)_{L^2(I_r^c)} \geq -\varepsilon(x, x)_{L^2}, \quad \forall x \in X_\beta^-, \quad (3.32)$$

where $I_r^c = \mathbb{R} \setminus I_r$, and from the definition of R_k ,

$$\nabla_z^2 R_k(t, z_k(t)) \geq B_\infty(t), \quad t \in I_r, \quad k \geq K(r), \quad (3.33)$$

that is

$$((\nabla_z^2 R_k(t, z_k) - B_\infty)x, x)_{L^2(I_r)} \geq 0, \quad \forall x \in X_\beta^-, \quad k \geq K(r). \quad (3.34)$$

From (3.32) and (3.34),

$$(\nabla_z^2 R_k(t, z_k)x, x)_{L^2} \geq ((B_\infty x, x)_{L^2} - \varepsilon(x, x)_{L^2}), \quad (3.35)$$

for k large enough. Thus we have

$$\begin{aligned} (a_k''(x_k)x, x)_{L^2} &= ((\tilde{F} - \nabla_z^2 R_k(t, z_k))x, x)_{L^2} - (\nabla_z^2(R_k(t, z_k))(z_k'^+ + z_k'^-)x, x)_{L^2} \\ &\leq ((\tilde{F} - B_\infty)x, x)_{L^2} + \frac{\alpha}{2}\|x\|_{L^2}^2 + \varepsilon\|x\|_{L^2}^2 \\ &\leq -\frac{\alpha}{4}\|x\|_{L^2}^2. \end{aligned} \quad (3.36)$$

That is $m_{a_k}^-(x) \geq m_{a_\infty}^-(0)$, from Theorem 2.5, $m_{a_k}^-(x_k) = \dim(E^-(\mathcal{H}_0)) + \mu_F(z_x(x_k))$, $m_{a_\infty}^-(0) = \dim(E^-(\mathcal{H}_0)) + \mu_F(KB_\infty)$, thus $\mu_F(z_x) \geq \mu_F(KB_\infty)$, which contradicts the assumption. \square

Proof of Theorem 1.3. As claimed in Remark 2.7, we can only consider the case of (R_∞^+) . Note that $z = 0$ is a critical point of a_k , the morse index of 0 for a_k is $m_{a_k}^-(0) = \dim(E^-(\mathcal{H}_0)) + \mu_F(KB_0)$, since $\gamma \cdot I_{2N} > B_\infty$, we have

$$\mu_F(K\gamma \cdot I_{2N}) \geq \mu_F(KB_\infty) \geq \mu_F(KB_0). \quad (3.37)$$

From proposition (2) in Lemma 3.4, use the $(m_{a_k}^-(0))^{th}$ and $(m_{a_k}^-(0) + 1)^{th}$ Morse inequalities, a_k has a nontrivial critical point x_k with its morse index $m_{a_k}^-(x_k) \leq m_{a_k}^-(0) + 1$, that is $\mu_F(z_k) \leq \mu_F(KB_0) + 1 \leq \mu_F(KB_\infty) - 1$, then from Lemma 3.5, we have $\{z_k\}$ is bounded in L^∞ . Thus z_k is a nontrivial solution of (HS) for k large enough. \square

The proof of Theorem 1.4 is similar to the proof of Theorem 1.3. Instead of Morse theory we make use of minimax arguments for multiplicity of critical points.

Let X be a Hilbert space and assume $\phi \in C^2(X, \mathbb{R})$ is an even functional, satisfying the (PS) condition and $\phi(0) = 0$. Denote $S_c = \{u \in X \mid \|u\| = c\}$.

Lemma 3.6. *(See [19, Corollary 10.19].) Assume Y and Z are subspaces of X satisfying $\dim Y = j > k = \text{codim} Z$. If there exist $R > r > 0$ and $\alpha > 0$ such that*

$$\inf \phi(S_r \cap Z) \geq \alpha, \quad \sup \phi(S_R \cap Y) \leq 0,$$

then ϕ has $j - k$ pairs of nontrivial critical points $\{\pm x_1, \pm x_2, \dots, \pm x_{j-k}\}$, so that $\mu(x_i) \leq k + i$, for $i = 1, 2, \dots, j - k$.

First, we consider the case of (R_∞^+) , since R is even, we have R_k is also even, and satisfies Lemma 3.3. Let $Y = X_\beta^-$, and Z the positive space of $a_k''(0)$ in X_β , and we have $\dim Y = E^-(X_\beta) + \mu_F(KB_\infty)$, $\text{codim} Z = E^-(X_\beta) + \mu_F(KB_0)$, $\dim Y > \text{codim} Z$. So a_k has $l := \mu_F(KB_\infty) - \mu_F(KB_0)$ pairs of nontrivial critical points

$$\{\pm x_1, \pm x_2, \dots, \pm x_l\},$$

and $l - 1$ pairs of them satisfy

$$m^-(x_i) \leq \mu_F(KB_0) + i < \mu_F(KB_\infty), \quad i = 1, 2, \dots, l - 1. \quad (3.38)$$

Then we can complete the proof. In order to prove the case of (R_∞^-) , we need the following lemma.

Lemma 3.7. *(See [8, Corollary II 4.1].) Assume Y and Z are subspaces of X satisfying $\dim Y = j > k = \text{codim} Z$. If there exist $r > 0$, and $\alpha > 0$ such that*

$$\inf \phi(Z) > -\infty, \quad \sup \phi(S_r \cap Y) \leq -\alpha,$$

then ϕ has $j - k$ pairs of nontrivial critical points $\pm u_1, \pm u_2, \dots, \pm u_{j-k}$ so that $\mu(u_i) + \nu(u_i) \geq k + i - 1$ for $i = 1, 2, \dots, j - k$.

The proof is similar to the case of (R_∞^+) , we omit it here.

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